TUTORIAL NOTES FOR MATH4220

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1. LOCAL PROPERTIES FOR SOLUTIONS OF THE PARABOLIC EQUATIONS

Let us investigate various properties for the solution of the heat equation. Denote $Q_R(t_0, x_0) := (t_0 - R^2, t_0] \times B_R(x_0)$ and $Q_R := Q_R(0, 0)$.

Example 1. Let $u \in C^{1,2}$ such that $\partial_t u - \Delta u = 0$ in $Q_R(t_0, x_0)$ for some $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ and $R > 0$. Then,

$$|D_x u(t_0, x_0)| \le \frac{C}{R} \|u\|_{L^1(Q_R(t_0,x_0))}.$$

In general, for $\beta \in \mathbb{N}$ and multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$,

$$|\partial_t^\beta D_x^\alpha u(t_0, x_0)| \le \frac{C_n^{|\alpha|+2|\beta|+1} e^{|\alpha|+2\beta-1}(|\alpha| + 2\beta)!}{R^{|\alpha|+2\beta}} \|u\|_{L^1(Q_R(t_0,x_0))},$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Proof. It suffices to consider the case for $(t_0, x_0) = (0, 0)$ and $\beta = 0$.

For $|\alpha| \le 1$. Let $(t, x) \in Q_{\frac{1}{4}R}$, choose a cutoff function $\varphi \in C^\infty(Q_R)$ with supp$\varphi \subset Q_{\frac{3}{4}R}$ and $\varphi \equiv 1$ in $Q_{\frac{1}{2}R}$, then set

$$v = \varphi(s, y)\Gamma(t - s, x - y),$$

where $\Gamma(t - s, x - y) = [4\pi(t - s)]^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} H(t - s)$. Since u is the solution of the heat equation, therefore

$$0 = v(\partial_s u - \Delta_y u) = \partial_s(uv) + \nabla_y \cdot (u\nabla_y v - v\nabla_y u) - u(\partial_s v + \Delta_y v).$$

For any $\varepsilon > 0$, integrate the above equation with respect to (s, y) in $(-R^2, t - \varepsilon) \times B_R$, then

$$\begin{aligned}
&\int_{B_R} \varphi(t - \varepsilon, y)u(t - \varepsilon, y)\Gamma(\varepsilon, x - y)dy \\
&= \int_{-R^2}^{t-\varepsilon} \int_{B_R} u(s, y)\partial_s(\varphi(s, y)\Gamma(t - s, x - y))dyds \\
&\quad + \int_{-R^2}^{t-\varepsilon} \int_{B_R} u(s, y)\Delta_y(\varphi(s, y)\Gamma(t - s, x - y))dyds,
\end{aligned}$$

let $\varepsilon \to 0$,

$$\begin{aligned}
u(t, x) = &\int_{-R^2}^{t} \int_{B_R} u(s, y)\partial_s(\varphi(s, y)\Gamma(t - s, x - y))dyds \\
&+ \int_{-R^2}^{t} \int_{B_R} \Delta_y(\varphi(s, y)\Gamma(t - s, x - y))dyds,
\end{aligned}$$

since $\partial_s\Gamma(t-s,x-y)+\Delta_y\Gamma(t-s,x-y)=0$ for $s<t$, therefore

$$u(t,x)=\int_{-R^2}^{0}\int_{B_R}u(s,y)(\partial_s\varphi(s,y)+\Delta_y\varphi(s,y))\Gamma(t-s,x-y))dyds$$
$$+\int_{-R^2}^{0}\int_{B_R}u(s,y)2\nabla_y\varphi(s,y)\cdot\nabla_y\Gamma(t-s,x-y)dyds.$$

Note that $\partial_s\varphi\equiv\nabla_y\varphi\equiv\Delta_y\varphi\equiv 0$ in $Q_{\frac{1}{2}R}$ and $Q_R/Q_{\frac{3}{4}R}$, then

$$\nabla_x u(t,x)=\int\int_{Q_{\frac{3}{4}R}/Q_{\frac{1}{2}R}}u(s,y)(\partial_s\varphi(s,y)+\Delta_y\varphi(s,y))\nabla_x\Gamma(t-s,x-y))dyds$$
$$+\int\int_{Q_{\frac{3}{4}R}/Q_{\frac{1}{2}R}}u(s,y)2\nabla_y\varphi(s,y)\cdot\nabla_x\nabla_y\Gamma(t-s,x-y)dyds.$$

Since there exists a constant $C>0$ such that

$$|\partial_s\varphi|+|\Delta_y\varphi|\le C,\quad |\nabla_y\varphi|\le C,$$

then

$$|\nabla_x u(t,x)|\le C\iint_{Q_{\frac{3}{4}R}/Q_{\frac{1}{2}R}}(|\nabla_x\Gamma(t-s,x-y)|+|\nabla_x\nabla_y\Gamma(t-s,x-y)|)|u(s,y)|dyds.$$

By the explicit expression for $\Gamma(t-s,x-y)$,

$$|\nabla_x\Gamma|\le C_n\frac{|x-y|}{(t-s)^{\frac{n}{2}+1}}e^{-\frac{|x-y|^2}{4(t-s)}},\quad |\nabla_x\nabla_y\Gamma|\le C_n\frac{|x-y|^2+(t-s)}{(t-s)^{\frac{n}{2}+2}}e^{-\frac{|x-y|^2}{4(t-s)}},$$

since for $(t,x)\in Q_{\frac{1}{4}R}$ and $(s,y)\in Q_{\frac{3}{4}R}/Q_{\frac{1}{2}R}$,

$$|x-y|\le R,\quad 0<t-s\le R,\quad \frac{1}{(t-s)^{\frac{n}{2}+i}}e^{-\frac{|x-y|^2}{4(t-s)}}\le C_n, i=1,2,$$

then

$$|\nabla_x u(t,x)|\le\frac{C_n^2}{R}\|u\|_{L^1(Q_R)},$$

for all $(t,x)\in Q_{\frac{1}{4}R}$, therefore

$$|\nabla_x u(0,0)|\le\frac{C_n^2}{R}\|u\|_{L^1(Q_R)}.$$

For $|\alpha|\ge 2$, we prove the result by induction. Suppose for $|\alpha|\le k-1$, it is valid that

$$|D_x^\alpha u(0,0)|\le\frac{C_n^{|\alpha|+1}e^{|\alpha|-1}|\alpha|!}{R^{|\alpha|}}\|u\|_{L^1(Q_R)},$$

where $|\alpha|=\alpha_1+\cdots+\alpha_n$. Now for arbitrary multi-index α with $|\alpha|=k$, there exist some $i\in\{1,...,n\}$ and multi-index α' with $|\alpha'|=k-1$ such that

$$D_x^\alpha u=\partial_{x_i}D_x^{\alpha'}u,$$

since $D_x^{\alpha'}u$ is also a solution of the heat equation, then by the similar discussion as above, for $0<\theta<1$,

$$|\partial_{x_i}D_x^{\alpha'}u(0,0)|\le\frac{C_n^2}{(1-\theta)R}\|D_x^{\alpha'}u\|_{L^1(Q_{(1-\theta)R})}.$$

Since for $(t,x) \in Q_{(1-\theta)R}$, $Q_{\theta R}(t,x) \subset Q_R$, therefore by the induction assumption,

$$|D^{\alpha'} u(t,x)| \le \frac{C_n^{|\alpha'|+1} e^{|\alpha'|-1} |\alpha'|!}{(1-\theta)\theta^{|\alpha'|} R^{|\alpha'|+1}} \|u\|_{L^1(Q_R)}.$$

Let $\theta = \frac{m}{m+1}$, then

$$\frac{1}{(1-\theta)\theta^{|\alpha'|}} < e(|\alpha'|+1),$$

therefore

$$|D_x^\alpha u(0,0)| \le \frac{C_n^{|\alpha|+1} e^{|\alpha|-1} |\alpha|!}{R^{|\alpha|}} \|u\|_{L^1(Q_R)}.$$

□

Example 2. Let $u \in C^{1,2}$ such that $\partial_t u - \Delta u = 0$ in $Q_R(t_0, x_0)$ for some $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ and $R > 0$, then for arbitrary $t \in (t_0 - R^2, t_0]$, $u(t, \cdot)$ is analytic in $B_R(x_0)$.

Proof. For arbitrary $t \in (t_0 - R^2, t_0]$, to prove $u(t, \cdot)$ is analytic in $B_R(x_0)$, we show $u(t, \cdot)$ can be represented by a convergent power series in a neighborhood of arbitrary $x' \in B_R(x_0)$.

Denote

$$r := \frac{1}{4} \mathrm{dist}(x', \partial B_R(x_0)).$$

For each $x \in B_r(x')$, we have $Q_r(t,x) \subset Q_{2r}(t,x')$, then

$$|D_x^\alpha u(t,x)| \le \frac{C_n^{|\alpha|+1} e^{|\alpha|-1} |\alpha|!}{(2r)^{|\alpha|}} \|u\|_{L^1(Q_{2r}(t,x'))}.$$

Consider the Taylor series for u at y_0,

$$S_N(t,x) = \sum_{|\alpha| \le N-1} \frac{D_x^\alpha u(t,x')}{\alpha!} (x - x')^\alpha,$$

we claim that $S_N(t,x)$ converges provided $x \in B_\varepsilon(x')$ with $0 < \varepsilon < \frac{r}{C_n e}$. Indeed, denote the remainder term

$$R_N(t,x) := u(t,x) - S_N(t,x) = \sum_{|\alpha|=N} \frac{D_x^\alpha u(t, x' + \theta(x-x'))(x-x')^\alpha}{\alpha!},$$

for some $0 \le \theta \le 1$. Then for $x \in B_\varepsilon(x')$,

$$\begin{aligned} |R_N(t,x)| \le & C \|u\|_{L^1(Q_{2r}(t,x'))} \sum_{|\alpha|=N} \frac{C_n^{N+1} e^{N-1}}{(2r)^N} \left(\frac{r}{C_n e}\right)^N \\ \le & C \|u\|_{L^1(Q_{2r}(t,x'))} \frac{C_n}{2^N e}, \end{aligned}$$

therefore by letting N goes to infinity, we have

$$\lim_{N \to \infty} |R_N(t,x)| = 0,$$

which implies that $S_N(t,x)$ converges to $u(t,x)$. □

Remark 3. The solution of the heat equation is not necessarily analytic in t. For example, consider

$$u(t,x) = \begin{cases} t^{-\frac{1}{2}} e^{-\frac{x^2}{4t}}, & x > 1, t > 0, \\ 0, & x > 1, t \le 0, \end{cases}$$

then u is a solution of the heat equation in $(-\infty, \infty) \times (1, \infty)$ but is not analytic in t.

A Supplementary Problem

Problem 4. Let $M \subset \mathbb{R}^{n+1}$ and $u \in C^{1,2}(M)$ is a solution of the heat equation. Let $(t_0, x_0) \in \partial M$ be the local maximal point of u. If there exists a ball $B_r(t_0, x_0) \subset M$ such that $\partial M \cap \overline{B_r(t_0, x_0)} = \{(t_0, x_0)\}$, then

$$\frac{\partial u}{\partial v}(t_0, x_0) > 0,$$

where $v = \frac{(t-t_0, x-x_0)}{r}$. In particular, if ∂M is C^1 at (t_0, x_0), then v is the inward unit normal to ∂M at the point (t_0, x_0).

For more materials, please refer to [1, 2, 3, 4].

References

[1] S. Alinhac, *Hyperbolic partial differential equations*, Universitext, Springer, Dordrecht, 2009.
[2] L. C. Evans, *Partial differential equations*, vol. 19 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 1998.
[3] Q. Han and F. Lin, *Elliptic partial differential equations*, vol. 1 of Courant Lecture Notes in Mathematics, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1997.
[4] W. A. Strauss, *Partial differential equations. An introduction*, John Wiley & Sons, Inc., New York, 1992.

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